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Theory and computation of spheroidal wavefunctions

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Abstract

In this paper we report on a package, written in the *Mathematica* computer algebra system, which has been developed to compute the spheroidal wavefunctions of Meixner and Schäfke (1954 *Mathieusche Funktionen und Sphäroidfunktionen*) and is available online (physics.uwa.edu.au/~falloon/spheroidal/spheroidal.html). This package represents a substantial contribution to the existing software, since it computes the spheroidal wavefunctions to arbitrary precision for general complex parameters μ, ν, γ and argument z; existing software can only handle integer μ, ν and does not give arbitrary precision. The package also incorporates various special cases and computes analytic power series and asymptotic expansions in the parameter γ. The spheroidal wavefunctions of Flammer (1957 *Spheroidal Wave functions*) are included as a special case of Meixner's more general functions. This paper presents a concise review of the general theory of spheroidal wavefunctions and a description of the formulae and algorithms used in their computation, and gives high precision numerical examples.

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1. Introduction

Spheroidal wavefunctions are a class of special functions with many applications in physics and applied mathematics. They satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((1-z^2)\frac{\mathrm{d}f}{\mathrm{d}z}\right)+\left(\lambda+\gamma^2(1-z^2)-\frac{\mu^2}{1-z^2}\right)f(z)=0 \qquad (1)$$

where μ, ν and γ are arbitrary complex parameters. For many applications μ, ν take on integer values, in which case they are denoted as m, n. Although the general theory of spheroidal wavefunctions has been known for a long time [1–3], they are still regarded as difficult to compute, and at present there are few readily available computer packages for their computation.

Solutions of (1) were first studied by Niven [4] in connection with a problem involving heat conduction in spheroidal bodies, and were subsequently investigated by a number of authors (see [5] and references therein). Early applications included the quantum mechanical two-centre problem and various electromagnetic boundary-value problems. The general theory and background on spheroidal wavefunctions is contained in the monograph by Flammer [2]. Other notable monographs are Stratton *et al* [3], which has extensive tables of numerical values, Komarov *et al* [6], Arscott [7] and Li [8]. These works focus exclusively on spheroidal wavefunctions with integer parameters m, n. Meixner [1, 9] developed the theory of spheroidal wavefunctions with arbitrary complex parameters μ, ν.

Several packages have been developed recently to compute spheroidal wavefunctions: Thompson [10] (which uses an incorrect expansion for the angular functions of the second kind) and Li *et al* [11] are two of the most recent. Both of these packages are only useful for small values of γ, do not provide arbitrary precision computation and are limited to integer parameters m, n. Furthermore, no package currently in existence computes the power series and asymptotic expansions for the spheroidal wavefunctions.

The choice of a suitable notation and normalization for spheroidal wavefunctions presents a significant challenge due to the large number of conventions in existence. The two main ones are those of Meixner [1] and Flammer [2] (also used in [12]). The latter is more commonly used in the literature, however, its use is usually limited to integer m, n. Indeed, Flammer's normalization scheme for the angular functions cannot readily be generalized to noninteger parameters, because it normalizes functions to a different constant depending on the parity of $n - m$. Meixner's notation, though less commonly used, has the fundamental advantage that it is suitable for (and was in fact developed specifically to handle) the case of general complex parameters.

The purpose of this paper is to describe a package which has recently been developed [14] to compute spheroidal wavefunctions with arbitrary parameter values, which overcomes the above-mentioned shortcomings of existing packages. We have chosen to use the *Mathematica* computer algebra system [13], which is ideal due to its symbolic and high precision numerical capabilities, as well as its large library of built-in special functions. We have taken a unique approach to the notation for the spheroidal functions, in that our package computes the functions of Flammer and Meixner as two distinct sets of functions. In this way, it is hoped that the package will be useful to as wide an audience as possible.

The layout of this paper is as follows. In section 2 we present a concise review of the theory of the spheroidal wavefunctions, starting with their definition as series of Legendre and spherical Bessel functions. We then discuss the important special case for the angular functions of the second kind when $\mu + \nu$ is an integer, before discussing the spheroidal joining factor which relates the angular and radial functions. In section 3 we describe the computation of the spheroidal wavefunctions, beginning with a discussion of the continued fraction and tridiagonal matrix methods used to compute the spheroidal eigenvalues. We then discuss the general numerical implementation of the spheroidal functions, and finally describe the approach used to generate the asymptotic and power series coefficients. We conclude with a discussion of some of the ways in which the accuracy of our numerical values can be verified. Four appendices are also included: appendix A contains definitions for the Legendre and spherical Bessel functions which are used in the package; appendix B contains formulae relating Flammer's spheroidal functions to those of Meixner; appendix C contains a summary of some important mathematical identities satisfied by the spheroidal wavefunctions; finally, in appendix D we present tables of sample function values to high precision, for the purposes of comparison with other packages.

2. Theory

2.1. The angular spheroidal wavefunctions

When $\gamma = 0$, (1) reduces to Legendre's differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((1-z^2)\frac{\mathrm{d}f}{\mathrm{d}z}\right)+\left(\lambda-\frac{\mu^2}{1-z^2}\right)f(z)=0. \tag{2}$$

The solutions to this equation are the (associated) Legendre functions of the first and second kind, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, with eigenvalue $\lambda = \nu(\nu+1)$. Traditionally, these functions are defined differently depending on whether or not z lies on the branch cut $z \in (-1, 1)$ (e.g., [12], sections 8.1–3). For many applications—particularly those involving computer algebra—this convention is inconvenient, since it precludes the use of identities which are valid for all z. An alternative approach which avoids this difficulty is to define two distinct types of Legendre function, each of which is valid for *all* values of z. The functions of type I, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, are equivalent to those usually defined on the $(-1, 1)$ cut. The functions of type II, which we denote as $\mathfrak{P}_\nu^\mu(z)$ and $\mathfrak{Q}_\nu^\mu(z)$, are equivalent to the functions usually defined for $z \notin (-1, 1)$. In appendix A we give the definitions and some important properties of these functions.

For nonzero γ, the angular spheroidal wavefunctions are defined as infinite series of the corresponding Legendre functions:

$$F_\nu^\mu(z;\gamma)=\sum_{k=-\infty}^{\infty}(-1)^k a_{\nu,2k}^\mu(\gamma) f_{\nu+2k}^\mu(z). \tag{3}$$

Here we follow Meixner's notation so that $F = \mathrm{ps}, \mathrm{qs}, \mathrm{Ps}, \mathrm{Qs}$ and $f = P, Q, \mathfrak{P}, \mathfrak{Q}$, respectively.

It can be readily shown [1] that the series coefficients $a_{\nu,2k}^\mu(\gamma)$ satisfy the three-term recurrence relation

$$A_{\nu,2k}^\mu(\gamma)a_{\nu,2k}^\mu(\gamma)+\left(B_{\nu,2k}^\mu(\gamma)-\lambda\right)a_{\nu,2k}^\mu(\gamma)+C_{\nu,2k}^\mu(\gamma)a_{\nu,2k}^\mu(\gamma)=0 \tag{4}$$

where

$$\begin{aligned}
A_{\nu,2k}^\mu(\gamma)&=-\gamma^2\frac{(\nu-\mu+2k-1)(\nu-\mu+2k)}{(2\nu+4k-3)(2\nu+4k-1)}\\
B_{\nu,2k}^\mu(\gamma)&=(\nu+2k)(\nu+2k+1)-2\gamma^2\frac{(\nu+2k)(\nu+2k+1)+\mu^2-1}{(2\nu+4k+3)(2\nu+4k+5)}\\
C_{\nu,2k}^\mu(\gamma)&=-\gamma^2\frac{(\nu+\mu+2k+1)(\nu+\mu+2k+2)}{(2\nu+4k+3)(2\nu+4k+5)}.
\end{aligned} \tag{5}$$

The series expansion (3) is convergent only when the coefficients $a_{\nu,2k}^\mu(\gamma)$ form a minimal solution to (4), (i.e. a solution with the property that $a_{\nu,2k\pm2}^\mu(\gamma)\big/a_{\nu,2k}^\mu(\gamma) \to 0$ as $k \to \pm\infty$ [18])—in which case it converges for all values of z. There is a countably infinite set of values of λ which correspond to minimal solutions. The *spheroidal eigenvalue* $\lambda_\nu^\mu(\gamma)$ is defined as a function of μ, ν and γ by choosing the minimal λ value which reduces to $\nu(\nu+1)$ continuously as $\gamma \to 0$ along the line $(0, \gamma)$ [1].

For integers m, n with $n \geqslant |m|$, the coefficients $a_{\nu,2k}^\mu(\gamma)$ are normalized so that

$$\int_{-1}^{1}\mathrm{ps}_n^m(t;\gamma)^2\,\mathrm{d}t=\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \tag{6}$$

which can be generalized in a natural way to give the relation for arbitrary μ, ν:

$$\sum_{k=-\infty}^{\infty}a_{\nu,2k}^\mu(\gamma)^2\frac{2\nu+1}{2\nu+4k+1}\frac{(\nu+\mu+1)_{2k}}{(\nu-\mu+1)_{2k}}=1. \tag{7}$$

The sign of $a^{\mu}_{\nu,2k}(\gamma)$ is determined by the condition that $\mathrm{ps}^{\mu}_{\nu}(z;\alpha\gamma) \to P^{\mu}_{\nu}(z)$ continuously as $\alpha \to 0$ along the line $0 \leqslant \alpha \leqslant 1$.

2.2. *Angular functions of the second kind for integer $\mu + \nu$*

The functions $Q^{\mu}_{\nu}(z)$ and $\mathfrak{Q}^{\mu}_{\nu}(z)$ diverge when $\mu+\nu$ approaches a negative integer (A.9), from which it immediately follows that:

$$\left|\mathrm{qs}^{\mu}_{\nu}(z;\gamma)\right| \qquad \left|\mathrm{Qs}^{\mu}_{\nu}(z;\gamma)\right| = \infty \qquad \text{for} \quad \mu+\nu = -1, -2, \ldots \tag{8}$$

For $\mu + \nu$ a *non*-negative integer, we have $C^{\mu}_{\nu,-\mu-\nu-2+\delta}(\gamma) = 0$, where $\delta = (\mu+\nu) \bmod 2$. From (4) we then find $a^{\mu}_{\nu,2k}(\gamma) = 0$ for $k < -(\mu+\nu)/2$, and hence the series (3) becomes indeterminate for $\mathrm{qs}^{\mu}_{\nu}(z;\gamma)$ and $\mathrm{Qs}^{\mu}_{\nu}(z;\gamma)$, since the infinite basis functions are multiplied by zero series coefficients. The procedure for recovering a valid series representation in this case is reasonably straightforward, and is described (for integer m, n) by Flammer [2].

The essential step is to use the transformation relations (A.10) and (A.11). Taking the case of $\mathfrak{Q}^{\mu}_{\nu}(z)$ for definiteness, we substitute $\nu \to \nu+\epsilon$, where $\mu+\nu$ is an integer and $\epsilon \to 0$, into (A.11) and immediately find

$$\begin{aligned}\sin(\epsilon\pi)\mathfrak{Q}^{\mu}_{\nu+\epsilon}(z) = (-1)^{\mu+\nu}\big(&\pi \mathrm{e}^{\mathrm{i}\mu\pi}\cos((\nu+\epsilon)\pi)\mathfrak{P}^{\mu}_{-\nu-\epsilon-1}(z)\\ &- \sin((\mu-\nu-\epsilon)\pi)\mathfrak{Q}^{\mu}_{-\nu-\epsilon-1}(z)\big).\end{aligned}$$

Multiplying by the series coefficient $a^{\mu}_{\nu+\epsilon,2k}(\gamma)$ and taking the limit $\epsilon \to 0$ we have

$$\begin{aligned}\lim_{\epsilon\to 0} a^{\mu}_{\nu+\epsilon,2k}(\gamma)\mathfrak{Q}^{\mu}_{\nu+2k+\epsilon}(z) &= \tilde{a}^{\mu}_{\nu,2k}(\gamma)(-1)^{\mu+\nu}\\ &\times\left(\mathrm{e}^{\mathrm{i}\mu\pi}\cos(\nu\pi)\mathfrak{P}^{\mu}_{-\nu-2k-1}(z) - \frac{\sin((\mu-\nu)\pi)}{\pi}\mathfrak{Q}^{\mu}_{-\nu-2k-1}(z)\right)\end{aligned}\tag{9}$$

where we define

$$\tilde{a}^{\mu}_{\nu,2k}(\gamma) = \lim_{\epsilon\to 0}\frac{a^{\mu}_{\nu+\epsilon,2k}(\gamma)}{\epsilon} \qquad k \leqslant k_0 - 1. \tag{10}$$

For $k \leqslant k_0 - 2$, the coefficients $\tilde{a}^{\mu}_{\nu,2k}(\gamma)$ can be computed using

$$\frac{\tilde{a}^{\mu}_{\nu,2k}(\gamma)}{\tilde{a}^{\mu}_{\nu,2k+2}(\gamma)} = -\frac{C^{\mu}_{\nu,2k}(\gamma)}{B^{\mu}_{\nu,2k}(\gamma) - \lambda^{\mu}_{\nu}(\gamma) + A^{\mu}_{\nu,2k}(\gamma)\frac{\tilde{a}^{\mu}_{\nu,2k-2}(\gamma)}{\tilde{a}^{\mu}_{\nu,2k}(\gamma)}} \tag{11}$$

while for $k = k_0 - 1$, we have

$$\tilde{a}^{\mu}_{\nu,2k_0-2}(\gamma) = -\frac{\tilde{C}^{\mu}_{\nu}(\gamma)a^{\mu}_{\nu,2k_0}(\gamma)}{B^{\mu}_{\nu,2k_0-2}(\gamma) - \lambda^{\mu}_{\nu}(\gamma) + A^{\mu}_{\nu,2k_0-2}(\gamma)\frac{\tilde{a}^{\mu}_{\nu,2k_0-4}(\gamma)}{\tilde{a}^{\mu}_{\nu,2k_0-2}(\gamma)}} \tag{12}$$

where

$$\tilde{C}^{\mu}_{\nu}(\gamma) = \lim_{\epsilon\to 0}\frac{C^{\mu}_{\nu+\epsilon,2k_0-2}(\gamma)}{\epsilon} = \frac{(-1)^{\delta}\gamma^2}{(2\mu-2\delta-1)(2\mu-2\delta+1)}. \tag{13}$$

For $\mu+\nu = 0, 1, 2, \ldots$ the series for $\mathrm{Qs}^{\mu}_{\nu}(z;\gamma)$ therefore reads

$$\begin{aligned}\mathrm{Qs}^{\mu}_{\nu}(z;\gamma) = \sum_{k=k_0}^{\infty}(-1)^k a^{\mu}_{\nu,2k}(\gamma)\mathfrak{Q}^{\mu}_{\nu+2k}(z) &+ (-1)^{\mu+\nu}\sum_{k=-\infty}^{k_0-1}(-1)^k\tilde{a}^{\mu}_{\nu,2k}(\gamma)\\ &\times\left(\mathrm{e}^{\mathrm{i}\mu\pi}\cos(\nu\pi)\mathfrak{P}^{\mu}_{-\nu-2k-1}(z) - \frac{\sin((\mu-\nu)\pi)}{\pi}\mathfrak{Q}^{\mu}_{-\nu-2k-1}(z)\right).\end{aligned}\tag{14}$$

For $\mathrm{qs}_\nu^\mu(z;\gamma)$ we follow a directly analogous argument, starting from (A.10) instead of (A.11), and obtain

$$\mathrm{qs}_\nu^\mu(z;\gamma) = \sum_{k=k_0}^{\infty}(-1)^k a_{\nu,2k}^{\mu}(\gamma) Q_{\nu+2k}^{\mu}(z) + (-1)^{\mu+\nu}\sum_{k=-\infty}^{k_0-1}(-1)^k \tilde{a}_{\nu,2k}^{\mu}(\gamma) \times\left(\cos(\mu\pi)\cos(\nu\pi)P_{-\nu-2k-1}^{\mu}(z) - \frac{\sin((\mu-\nu)\pi)}{\pi}Q_{-\nu-2k-1}^{\mu}(z)\right). \quad (15)$$

For integer m, n these expressions reduce to

$$\mathrm{Qs}_n^m(z;\gamma) = \sum_{k=k_0}^{\infty}(-1)^k a_{n,2k}^{m}(\gamma)\mathfrak{Q}_{n+2k}^{m}(z) + \sum_{k=-\infty}^{k_0-1}(-1)^k \tilde{a}_{n,2k}^{m}(\gamma)\mathfrak{P}_{-n-2k-1}^{m}(z) \quad (16)$$

$$\mathrm{qs}_n^m(z;\gamma) = \sum_{k=k_0}^{\infty}(-1)^k a_{n,2k}^{m}(\gamma) Q_{n+2k}^{m}(z) + \sum_{k=-\infty}^{k_0-1}(-1)^k \tilde{a}_{n,2k}^{m}(\gamma) P_{-n-2k-1}^{m}(z). \quad (17)$$

2.3. The radial spheroidal wavefunctions

If we let $\zeta = \gamma z$ and substitute $f(z) = (1-1/z^2)^{\mu/2} g(\zeta)$ into (1) we obtain, in the limit $\gamma \to 0$,

$$\zeta^2\frac{\mathrm{d}^2 g}{\mathrm{d}\zeta^2} + 2\zeta\frac{\mathrm{d}g}{\mathrm{d}\zeta} + (\zeta^2 - \lambda)g(\zeta) = 0. \quad (18)$$

Solutions of this equation are the spherical Bessel functions $j_\nu(\zeta)$ and $y_\nu(\zeta)$ (appendix A, [12] ch 10). The radial spheroidal wavefunctions are defined in terms of these by the series

$$S_\nu^{\mu(i)}(z;\gamma) = \frac{(1-1/z^2)^{\mu/2}}{A_\nu^{-\mu}(\gamma)}\sum_{k=-\infty}^{\infty} a_{\nu,2k}^{-\mu}(\gamma) f_{\nu+2k}(\gamma z) \quad (19)$$

where $i = 1, 2$ and $f = j, y$, respectively, and

$$A_\nu^\mu(\gamma) = \sum_{k=-\infty}^{\infty}(-1)^k a_{\nu,2k}^{\mu}(\gamma). \quad (20)$$

It has been shown [15] that the functions defined in (19) are indeed solutions of (1) which are absolutely convergent for $|z| > 1$. The normalization factor $A_\nu^\mu(\gamma)$ is chosen so that the radial functions have the following behaviour as $\gamma z \to \infty$:

$$S_\nu^{\mu(1)}(z;\gamma) \longrightarrow \frac{1}{\gamma z}\sin\left(\gamma z - \frac{\nu\pi}{2}\right) \qquad S_\nu^{\mu(2)}(z;\gamma) \longrightarrow -\frac{1}{\gamma z}\cos\left(\gamma z - \frac{\nu\pi}{2}\right).$$

The functions $S_\nu^{\mu(1,2)}(z;\gamma)$ have branch cuts in the complex z-plane along the semi-infinite line starting at the point $z = 0$ and passing through $z = -1/\gamma$, for noninteger ν, and on the interval $(-1, 1)$, for noninteger $\mu/2$.

2.4. Joining relations between angular and radial functions

The angular and radial spheroidal wavefunctions can both be considered as functions over the entire complex z-plane. From a computational point of view, however, their series expansions are only useful over a restricted subset of the complex plane. For the angular functions, the series (3) is convergent over the entire z-plane, but for $|z| > 1$ it becomes too slowly convergent to be of any practical use. For the radial functions the situation is even worse: the series (19) is in general not convergent inside the unit circle $|z| < 1$. To allow computation of the functions over the entire complex plane, Meixner and Schäfke [1] introduced a joining factor which relates the angular and radial functions.

Using well-known series expansions for $\mathfrak{Q}_\nu^\mu(z)$ and $j_\nu(z)$ (equations (A.8) and (A.12)), it is possible to find the following series expansions for $\mathrm{Qs}_\nu^\mu(z;\gamma)$ and $S_\nu^{\mu\,(1)}(z;\gamma)$ [14]:

$$\mathrm{Qs}_\nu^\mu(z;\gamma) = 2^\nu\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\mu\pi} z^{\nu-\mu}(z-1)^{\mu/2}(z+1)^{\mu/2} \times \sum_{j=-\infty}^{\infty}\frac{1}{(2z)^{2j}}\sum_{k=0}^{\infty}\frac{(-1)^{j-k}\Gamma(2j+\mu-\nu)a_{\nu,-2j+2k}^\mu(\gamma)}{\Gamma\left(2j-k-\nu+\frac{1}{2}\right)k!} \tag{21}$$

$$S_\nu^{\mu\,(1)}(z;\gamma) = \frac{\sqrt{\pi}(1-1/z^2)^{\mu/2}(\gamma z)^\nu}{2^{\nu+1}A_\nu^{-\mu}(\gamma)}\sum_{j=-\infty}^{\infty}\frac{1}{(2z)^{2j}}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{4j} a_{\nu,-2j-2k}^{-\mu}(\gamma)}{\Gamma\left(-2j-k+\nu+\frac{3}{2}\right)k!\gamma^{2j}}. \tag{22}$$

Comparison of (21) and (22) reveals that the expansions of $\mathrm{Qs}_{-\nu-1}^\mu(z;\gamma)$ and $S_\nu^{\mu\,(1)}(z;\gamma)$ involve identical powers of z. Furthermore, it is obvious from (A.13) that the expansion for $S_\nu^{\mu\,(2)}(z;\gamma)$ will *not* involve the same powers of z. Now, $\mathrm{Qs}_{-\nu-1}^\mu(z;\gamma)$ must be expressible as a linear combination of the functions $S_\nu^{\mu\,(1,2)}(z;\gamma)$, since it satisfies the same differential equation, so it follows that $\mathrm{Qs}_{-\nu-1}^\mu(z;\gamma)$ and $S_\nu^{\mu\,(1)}(z;\gamma)$ must be equal up to a constant factor. Comparing (21) and (22), we then have that the ratio

$$\left(\sum_{k=0}^{\infty}\frac{(-1)^k 2^{4j} a_{\nu,-2j-2k}^{-\mu}(\gamma)}{\Gamma\left(-2j-k+\nu+\frac{3}{2}\right)k!\gamma^{2j}}\right)\Bigg/\left(\sum_{k=0}^{\infty}\frac{(-1)^{j-k}\Gamma(2j+\mu-\nu)a_{\nu,-2j+2k}^\mu(\gamma)}{\Gamma\left(2j-k-\nu+\frac{1}{2}\right)k!}\right)$$

must be independent of j.

In the light of the above result, we define the *spheroidal joining factor* by the relation

$$S_\nu^{\mu\,(1)}(z;\gamma) = K_\nu^\mu(\gamma)\frac{\sin((\mu-\nu)\pi)}{\pi}\frac{\mathrm{e}^{-\mathrm{i}(\mu+\nu)\pi}(1-1/z^2)^{\mu/2}(\gamma z)^\nu}{\gamma^\nu z^{\nu-\mu}(z-1)^{\mu/2}(z+1)^{\mu/2}}\mathrm{Qs}_{-\nu-1}^\mu(z;\gamma). \tag{23}$$

The trigonometric and exponential factors are included so that the joining factor relations reduce to a simple form for integer m, n. Note also the factor involving various powers of z, which includes the branch cut information for the two functions. Comparing terms in (21) and (22) for any particular j we can obtain an explicit definition for $K_\nu^\mu(\gamma)$. For definiteness we choose $j = 0$ and obtain

$$K_\nu^\mu(\gamma) = \mathrm{e}^{\mathrm{i}\nu\pi}2^{-2\nu-1}\Gamma(\nu-\mu+1)\frac{\gamma^\nu}{A_\nu^{-\mu}(\gamma)} \times\left(\sum_{k=0}^{\infty}\frac{(-1)^k a_{\nu,-2k}^{-\mu}(\gamma)}{\Gamma\left(-k+\nu+\frac{3}{2}\right)k!}\right)\Bigg/\left(\sum_{k=0}^{\infty}\frac{a_{\nu,2k}^\mu(\gamma)}{\Gamma\left(-k-\nu+\frac{1}{2}\right)k!}\right). \tag{24}$$

Using this equation and the symmetry relations given in appendix C it is possible to obtain

$$A_\nu^{-\mu}(\gamma)K_{-\nu-1}^\mu(\gamma)A_\nu^\mu(\gamma)K_\nu^{-\mu}(\gamma) = \frac{\pi}{\gamma\sin((\mu+\nu)\pi)} \tag{25}$$

which is useful in constructing joining relations for integer m, n.

3. Numerical computation

3.1. The spheroidal eigenvalues $\lambda_\nu^\mu(\gamma)$

As we mentioned in section 2.1, the spheroidal eigenvalues $\lambda_\nu^\mu(\gamma)$ are minimal solutions of the three-term recurrence (4). There are two standard procedures for finding such solutions.

The first was developed independently by Bouwkamp [16] and Blanch [17], and makes use of a fundamental equivalence between three-term recurrences and continued fractions [18]. This provides a method for determining the eigenvalues numerically to high precision, although it relies on the availability of a sufficiently accurate starting estimate for the eigenvalue. The second method, due to Hodge [19], involves expressing the three-term recurrence as an infinite tridiagonal matrix equation. It is complementary to the first in the sense that it provides an excellent method for generating accurate starting estimates for the eigenvalues, but is inefficient for obtaining high precision eigenvalues. We now discuss both methods in turn.

3.1.1. Continued fraction method: Defining

$$\begin{aligned}
\alpha^{\mu}_{\nu,2k}(\gamma) &= A^{\mu}_{\nu,2k}(\gamma)C^{\mu}_{\nu,2k-2}(\gamma)\\
\beta^{\mu}_{\nu,2k}(\gamma) &= B^{\mu}_{\nu,2k}(\gamma)\\
N^{\mu}_{\nu,2k}(\gamma) &= C^{\mu}_{\nu,2k-2}(\gamma)\frac{a^{\mu}_{\nu,2k}(\gamma)}{a^{\mu}_{\nu,2k-2}(\gamma)}
\end{aligned} \tag{26}$$

equation (4) can be rewritten in ascending and descending form as

$$\begin{aligned}
N^{\mu}_{\nu,2k+2}(\gamma) &= \frac{\alpha^{\mu}_{\nu,2k+2}(\gamma)}{\beta^{\mu}_{\nu,2k+2}(\gamma) - \lambda^{\mu}_{\nu}(\gamma) - N^{\mu}_{\nu,2k+2}(\gamma)}\\
&= \beta^{\mu}_{\nu,2k}(\gamma) - \lambda^{\mu}_{\nu}(\gamma) - \frac{\alpha^{\mu}_{\nu,2k}(\gamma)}{N^{\mu}_{\nu,2k}(\gamma)}.
\end{aligned}$$

Setting $k = 0$ and iterating these relations we obtain

$$\mathcal{U}^{\mu(1)}_{\nu}(\gamma,\lambda) + \mathcal{U}^{\mu(2)}_{\nu}(\gamma,\lambda) = 0 \tag{27}$$

where we have defined

$$\begin{aligned}
\mathcal{U}^{\mu(1)}_{\nu}(\gamma,\lambda) &= \beta_0 - \lambda - \frac{\alpha_0}{\beta_{-2} - \lambda -}\,\frac{\alpha_{-2}}{\beta_{-4} - \lambda -}\cdots\\
\mathcal{U}^{\mu(2)}_{\nu}(\gamma,\lambda) &= -\frac{\alpha_2}{\beta_2 - \lambda -}\,\frac{\alpha_4}{\beta_4 - \lambda -}\,\frac{\alpha_6}{\beta_6 - \lambda -}\cdots
\end{aligned} \tag{28}$$

Here we are using a standard notational convention for continued fractions [20]:

$$\frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\frac{a_3}{b_3 +}\cdots = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}. \tag{29}$$

Equation (27) is a transcendental equation in λ, whose roots are the spheroidal eigenvalues $\lambda^{\mu}_{\nu+2k}(\gamma)$. The method of Bouwkamp and Blanch consists of differentiating the left side of (27) and using Newton's method. Differentiating (27) with respect to λ we obtain

$$\begin{aligned}
\frac{\partial\mathcal{U}^{\mu(1)}_{\nu}(\gamma,\lambda)}{\partial\lambda} &= -\left(1 + \frac{\alpha_0}{N_0^2} + \frac{\alpha_0}{N_0^2}\frac{\alpha_{-2}}{N_{-2}^2} + \frac{\alpha_0}{N_0^2}\frac{\alpha_{-2}}{N_{-2}^2}\frac{\alpha_{-4}}{N_{-4}^2} + \cdots\right)\\
\frac{\partial\mathcal{U}^{\mu(2)}_{\nu}(\gamma,\lambda)}{\partial\lambda} &= -\left(\frac{N_1^2}{\alpha_1} + \frac{N_2^2}{\alpha_2}\frac{N_4^2}{\alpha_4} + \frac{N_2^2}{\alpha_2}\frac{N_4^2}{\alpha_4}\frac{N_6^2}{\alpha_6} + \cdots\right).
\end{aligned} \tag{30}$$

To apply Newton's method to (27) we begin with a starting estimate λ_0 and iterate Newton's formula,

$$\delta\lambda_i = -\frac{\mathcal{U}^{\mu(1)}_{\nu}(\gamma,\lambda_{i-1}) + \mathcal{U}^{\mu(2)}_{\nu}(\gamma,\lambda_{i-1})}{\frac{\partial\mathcal{U}^{\mu(1)}_{\nu}(\gamma,\lambda_{i-1})}{\partial\lambda} + \frac{\partial\mathcal{U}^{\mu(2)}_{\nu}(\gamma,\lambda_{i-1})}{\partial\lambda}} \tag{31}$$

starting with $i = 1$, until the size of the ith iterate $\delta\lambda_i$ is beneath the desired level of precision. Provided sufficient precision is used throughout the application of this algorithm, it can be used to obtain $\lambda_\nu^\mu(\gamma)$ to arbitrarily high accuracy. Usually the intermediate working precision needs to be significantly higher than the desired final precision, as precision is almost always lost in the process of numerical computation.

3.1.2. Tridiagonal matrix method: In order to use the continued fraction method just described, it is necessary to begin with a reasonably accurate starting value. Traditionally, most authors have used a power series expansion of $\lambda_\nu^\mu(\gamma)$ in powers of γ [2], which has a very small radius of convergence (approximately between 4 and 5). Hodge [19] appears to have been the first to solve (4) by recasting it as a tridiagonal matrix equation:

$$\begin{pmatrix} \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & \\ & A_{-2} & B_{-2} & C_{-2} & & & \\ & & A_0 & B_0 & C_0 & & \\ & & & A_2 & B_2 & C_2 & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ a_{-2} \\ a_0 \\ a_2 \\ \cdot \\ \cdot \end{pmatrix} = \lambda_\nu^\mu(\gamma) \begin{pmatrix} \cdot \\ \cdot \\ a_{-2} \\ a_0 \\ a_2 \\ \cdot \\ \cdot \end{pmatrix}. \tag{32}$$

Truncating this equation in both directions yields a finite matrix, the eigenvalues of which will be approximations to the actual eigenvalues $\lambda_\nu^\mu(\gamma)$. Since eigenvalues of tridiagonal matrices can be evaluated very efficiently, this method allows approximate eigenvalues to be generated rapidly. These matrix eigenvalues are ideal starting estimates for the continued fraction method described above. Surprisingly, to our knowledge no author seems previously to have used this hybrid approach.

One issue which presents a practical challenge to the computation of the spheroidal eigenvalues is actually deciding which matrix eigenvalue corresponds to which value of ν: all that can be said in general is that each eigenvalue of the matrix corresponds to $\lambda_{\nu+2k}^\mu(\gamma)$ for some integer k. When γ^2 is real and m, n are integers, the eigenvalues are strictly ordered and hence the correspondence is trivial. However, in the complex γ-plane the eigenvalues have a complicated branch cut structure, and the ordering relation does not apply. The same is true for noninteger μ, ν. The only available method which is fully rigorous is to iteratively compute eigenvalues $\lambda_\nu^\mu(\alpha\gamma)$ for values of α lying along the line segment $0 \leqslant \alpha \leqslant 1$, starting at $\alpha = 0$. Provided the step sizes are small enough, each successive eigenvalue can be used to choose the correct starting value from the matrix at the subsequent step. Unfortunately, this approach leads to an intractable amount of computation unless parameter values are very small. In the package, therefore, we simply include an optional extra argument to the eigenvalue function, which allows a starting estimate to be given by the user. Using this extra argument, it is straightforward to implement the above iterative procedure for particular cases.

3.2. *The spheroidal wavefunctions*

Once the eigenvalue $\lambda_\nu^\mu(\gamma)$ has been found, it is relatively straightforward to compute the actual spheroidal wavefunctions using (3) and (19). There are two main parts to this computation: generating the series coefficients $a_{\nu,2k}^\mu(\gamma)$ and generating the basis functions (Legendre or spherical Bessel functions). We now discuss each of these in turn.

3.2.1. Series coefficients: To generate the series coefficients $a^{\mu}_{\nu,2k}(\gamma)$ we follow the standard approach of rewriting the recurrence relation (4) in terms of ascending and descending ratios:

$$\begin{aligned}\frac{a^{\mu}_{\nu,2k}(\gamma)}{a^{\mu}_{\nu,2k+2}(\gamma)} &= -\frac{C^{\mu}_{\nu,2k}(\gamma)}{B^{\mu}_{\nu,2k}(\gamma) - \lambda^{\mu}_{\nu}(\gamma) + A^{\mu}_{\nu,2k}(\gamma)\frac{a^{\mu}_{\nu,2k-2}(\gamma)}{a^{\mu}_{\nu,2k}(\gamma)}}\\ \frac{a^{\mu}_{\nu,2k}(\gamma)}{a^{\mu}_{\nu,2k-2}(\gamma)} &= -\frac{A^{\mu}_{\nu,2k}(\gamma)}{B^{\mu}_{\nu,2k}(\gamma) - \lambda^{\mu}_{\nu}(\gamma) + C^{\mu}_{\nu,2k}(\gamma)\frac{a^{\mu}_{\nu,2k+2}(\gamma)}{a^{\mu}_{\nu,2k}(\gamma)}}.\end{aligned} \tag{33}$$

These ratios converge as $1/k^2$ as $k \to \pm\infty$, so we can set them approximately to zero for some large value $|k| = k_{\max}$. We then iterate (33) from $k = \pm k_{\max}$ to $k = 0$ to obtain $a^{\mu}_{\nu,2k}(\gamma)\big/a^{\mu}_{\nu,0}(\gamma)$ for $-k_{\max} \leqslant k \leqslant k_{\max}$. Once again, in this recursive process precision is usually lost with each step, and hence it is necessary to work with an intermediate precision substantially greater than the final desired precision. In practice, our package begins with a working precision of 100 extra digits, then tests at the end whether the final precision is high enough. If it is not, the working precision is increased by 100 and the coefficients computed again.

The normalization relation (7) can be rewritten in the form

$$a^{\mu}_{\nu,0}(\gamma) \simeq \left(\sum_{k=-k_{\max}}^{k_{\max}} \left(\frac{a^{\mu}_{\nu,2k}(\gamma)}{a^{\mu}_{\nu,0}(\gamma)}\right)^2 \frac{2\nu+1}{2\nu+4k+1}\frac{(\nu+\mu+1)_{2k}}{(\nu-\mu+1)_{2k}}\right)^{-1/2} \tag{34}$$

allowing us to obtain $a^{\mu}_{\nu,0}(\gamma)$ and hence the correctly normalized list of coefficients $a^{\mu}_{\nu,2k}(\gamma)$ for $-k_{\max} \leqslant k \leqslant k_{\max}$.

The question is now whether or not the chosen value of $k_{\max}$ is large enough that the omitted 'tail' of the series is insignificant to the given level of precision. To answer this question rigorously is not easy, and in the package we settle for simply checking that the magnitude of $a^{\mu}_{\nu,\pm 2k}(\gamma)$ is negligible compared to the largest series coefficient, i.e.

$$\frac{\left|a^{\mu}_{\nu,\pm 2k_{\max}}(\gamma)\right|}{\max\left(\left|a^{\mu}_{\nu,2k}(\gamma)\right| : -k_{\max} \leqslant k \leqslant k_{\max}\right)} < 10^{-p} \tag{35}$$

where p represents the number of digits of precision required. Although not mathematically rigorous, extensive numerical testing shows that this is an eminently reasonable condition.

3.2.2. Basis functions: In principle, the basis functions could be computed using *Mathematica*'s built-in functions. However, it is much more efficient to start with the basis functions with $k = 0, 1$ and then use recurrence relations satisfied by the basis functions to generate the basis functions for all other values of k. The required relations are

$$\begin{aligned}&\frac{(\mu-\nu-2)(\mu-\nu-1)}{(2\nu+1)(2\nu+3)} f^{\mu}_{\nu+2}(z) + \left(\frac{2\nu(\nu+1)-2\mu^2-1}{(2\nu-1)(2\nu+3)} - z^2\right) f^{\mu}_{\nu}(z)\\ &\qquad + \frac{(\mu+\nu-1)(\mu+\nu)}{(2\nu-1)(2\nu+1)} f^{\mu}_{\nu-2}(z) = 0\end{aligned} \tag{36}$$

for $f = P, Q, \mathfrak{P}, \mathfrak{Q}$, and

$$\frac{f_{\nu-2}(z)}{2\nu-1} + (2\nu+1)\left(\frac{2}{(2\nu-1)(2\nu+3)} - \frac{1}{z^2}\right) f_{\nu}(z) + \frac{f_{\nu+2}(z)}{2\nu+3} = 0 \tag{37}$$

for $f = j, y$. These relations are readily obtained from the standard recurrence formulae (8.5.3) and (10.1.19) in [12]. Once again, these computations must be carried out at a much higher precision than is required at the end.

3.3. Series expansions

The eigenvalues $\lambda_\nu^\mu(\gamma)$ and the ratios $a_{\nu,2k}^\mu(\gamma)\big/a_{\nu,0}^\mu(\gamma)$ can be expanded in powers of γ^2 [2], leading to approximations which are useful for $|\gamma| \leqslant 4$–5. However, for these values of γ the tridiagonal matrix approach of section 3.1 is much more useful for obtaining numerical values, so in practice the power series is mainly of theoretical interest. We now describe how the power series expansions are computed in our package.

We begin with the following *ansätze*:

$$\lambda_\nu^\mu(\gamma) = \sum_{j=0}^{\infty} l_j^{\mu\nu}\gamma^{2j} \tag{38}$$

$$\frac{a_{\nu,2k}^\mu(\gamma)}{a_{\nu,0}^\mu(\gamma)} = \sum_{j=0}^{\infty} \alpha_{j,2k}^{\mu\nu}\gamma^{2j} \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{39}$$

By inspection we immediately have

$$\alpha_{0,0}^{\mu\nu} = 1 \qquad \alpha_{j,0}^{\mu\nu} = \alpha_{0,2k}^{\mu\nu} = 0 \qquad \text{for} \quad j, k \neq 0. \tag{40}$$

Substituting (38) and (39) into the recurrence relation (4) with $k = 0$, and defining

$$\begin{aligned} \mathcal{A}_{\nu,2k}^\mu &= \frac{(\mu+\nu+2k+1)(\mu+\nu+2k+2)}{(2\nu+4k+3)(2\nu+4k+5)} \\ \mathcal{B}_{\nu,2k}^\mu &= \frac{1}{2}\left(1 - \frac{4\mu^2-1}{(2\nu+4k-1)(2\nu+4k+3)}\right) \\ \mathcal{C}_{\nu,2k}^\mu &= \frac{(\nu-\mu+2k)(\nu-\mu+2k-1)}{(2\nu+4k-3)(2\nu+4k-1)} \end{aligned} \tag{41}$$

we find

$$\frac{\nu(\nu+1) - l_0^{\mu\nu}}{\gamma^2} + \mathcal{B}_{\nu,0}^\mu - l_1^{\mu\nu} + \sum_{j=1}^{\infty}\left(\mathcal{A}_{\nu,0}^\mu\alpha_{j,2}^{\mu\nu} + \mathcal{C}_{\nu,0}^\mu\alpha_{j,-2}^{\mu\nu} - l_{j+1}^{\mu\nu}\right)\gamma^{2j} = 0. \tag{42}$$

Since this is true for all values of γ it follows that

$$\begin{aligned} l_0^{\mu\nu} &= \nu(\nu+1) \\ l_1^{\mu\nu} &= \mathcal{B}_{\nu,0}^\mu = \frac{1}{2}\left(1 - \frac{4\mu^2-1}{(2\nu-1)(2\nu+1)}\right) \\ l_j^{\mu\nu} &= \mathcal{A}_{\nu,0}^\mu\alpha_{j-1,2}^{\mu\nu} + \mathcal{C}_{\nu,0}^\mu\alpha_{j-1,-2}^{\mu,\nu} \qquad \text{for} \quad j = 2, 3, \ldots. \end{aligned} \tag{43}$$

This provides a recursive method for determining $l_j^{\mu\nu}$ when the coefficients $\alpha_{j-1,\pm 2}^{\mu\nu}$ are known. To find these coefficients we substitute (38) and (39) into (4) for general k. After some manipulation we find

$$\begin{aligned} \alpha_{j,2k}^{\mu\nu} = \frac{1}{(\nu+2k)(\nu+2k+1) - l_0^{\mu\nu}} & \left(\sum_{i=0}^{j-1} l_{i+1}^{\mu\nu}\alpha_{j-i-1,2k}^{\mu\nu}\right. \\ & \left. - \left(\mathcal{A}_{\nu,2k}^\mu\alpha_{j-1,2k+2}^{\mu\nu} + \mathcal{B}_{\nu,2k}^\mu\alpha_{j-1,2k}^{\mu\nu} + \mathcal{C}_{\nu,2k}^\mu\alpha_{j-1,2k-2}^{\mu\nu}\right)\right). \end{aligned} \tag{44}$$

Now let k be a positive integer and suppose that $\alpha_{j-1,2k-2}^{\mu\nu} = 0$ and $\alpha_{r,2s}^{\mu\nu} = 0$ for all $r \leqslant j-1$ and $s > k$. Then (44) shows that $\alpha_{j,2k}^{\mu\nu} = 0$. This fact, together with the initial conditions $\alpha_{0,2k}^{\mu\nu} = 0$ for $k = \pm 1, \pm 2, \ldots$, proves by induction that

$$\alpha_{j,2k}^{\mu\nu} = 0 \qquad \text{for} \quad j < |k| \tag{45}$$

and hence $a^{\mu}_{\nu,\pm 2k}(\gamma)\big/a^{\mu}_{\nu,0}(\gamma) \sim O(\gamma^{2k})$ for $k = 0, 1, 2, \ldots$ This result could also have been deduced directly from (4). Equations (43)–(45) constitute the recursive scheme by which we compute the power series expansions (38) and (39).

The key to an efficient implementation of recursive algorithms of this kind is to use *dynamic programming*, whereby coefficients are 'cached' once they are generated, allowing subsequent coefficients to be generated more quickly. In *Mathematica*, this is achieved with a definition of the form $f(x\_) := f(x) = \cdots$.

With the expansions for $a^{\mu}_{\nu,2k}(\gamma)\big/a^{\mu}_{\nu,0}(\gamma)$ computed, it is not difficult to obtain the corresponding expansions for the angular functions themselves. One extra step is required, however, since the coefficient $a^{\mu}_{\nu,0}(\gamma)$ is itself a function of γ and so must be expanded as well. This is readily accomplished using the relation

$$a^{\mu}_{\nu,0}(\gamma) = \left(\sum_{k=-\infty}^{\infty}\left(\frac{a^{\mu}_{\nu,2k}(\gamma)}{a^{\mu}_{\nu,0}(\gamma)}\right)^2 \frac{2\nu+1}{2\nu+4k+1}\frac{(\nu+\mu+1)_{2k}}{(\nu-\mu+1)_{2k}}\right)^{-1/2}.$$

For large γ, asymptotic series expansions are also known for the spheroidal wavefunctions. For integer m, n the angular functions reduce to Hermite/Laguerre polynomials as $\gamma^2 \to \pm\infty$, and asymptotic expansions in descending powers of γ can be found [2]. In our package, these asymptotic expansions are computed using the method just described for the power series.

4. Results and discussion

The most comprehensive sources of tabulated values of spheroidal functions are Flammer [2], Stratton *et al* [3] and Van Buren *et al* [21]. Some of the tables from Flammer are reproduced in tables 21.1–4 of Abramowitz and Stegun [12]. Some minor errors in these works have already been pointed out by Li *et al* [11]. We have compared the output of our package with all of these sources, and found agreement up to the precision to which the tabulated values are given.

With the availability of packages such as the one we have developed, there is clearly little need for exhaustive tabulations of numerical values. However, it is useful to give some high precision numerical values to provide a benchmark for comparison with other programs. In appendix D we provide a set of such values. They are presented to 25 digits and are intended only to represent an illustrative sample.

Because no tables or software packages presently available are capable of generating results to arbitrary precision, the most reliable way to check the validity of our numerical functions is to perform self-consistency tests. Although there are relatively few analytic results available for the spheroidal wavefunctions, there are several important tests (each of which we have applied to our package with perfect results):

- *Exact solutions*: for $n = 1, 2, \ldots$ we have $\lambda^1_n(n\pi/2) = 0$ [2]. Also, for $\mu = \frac{1}{2}$ the spheroidal functions reduce to the Mathieu functions ([12], ch 20), and the eigenvalues are related to the Mathieu characteristic values $a_r(q)$ by $\lambda^{\frac{1}{2}}_{\nu}(\gamma) = a_{\nu+\frac{1}{2}}(\gamma^2/4) - \gamma^2/2 - \frac{1}{4}$ [14]. We can therefore compare the numerical eigenvalues generated by our package to the built-in Mathieu functions in *Mathematica*.
- *Wronskian*: for the spheroidal functions the Wronskian is proportional to $(z^2-1)^{-1}$, and for the radial functions it is equal to $(\gamma(z^2-1))^{-1}$. This test is the most generally useful, since it can be used for all parameter values.
- *Substitution into the differential equation*: it is straightforward to simply substitute the functions back into (1) and verify that it is satisfied to the precision of the numerical

functions. However, because the second derivative has to be computed numerically, this is not very convenient for testing to a very high level of precision.

- Lastly, a simple check that can always be performed is to generate a certain function value to two different levels of precision (e.g., 50 and 100 digits). If the two results do not agree up to the precision of the least precise of the two, this indicates there is a problem in the method of calculation. If they do agree, however, it does not guarantee anything—since, for example, the truncated series from which the function is being computed may contain too few terms—but it is a useful guide.

Our numerical package, Spheroidal.m, is available online at the URL physics.uwa.edu.au/~falloon/spheroidal/spheroidal.html, along with further documentation regarding its use.

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Appendix A. Legendre and spherical Bessel functions

In this appendix we give the definitions of the Legendre and spherical Bessel functions used to define the spheroidal wavefunctions. The definitions which we use here differ slightly from those found in [12], and are based on the approach taken in [22]. In particular, we define two types of Legendre function: the functions of type I, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, are equal to those conventionally used on the interval $z \in (-1, 1)$; the functions of type II, $\mathfrak{P}_\nu^\mu(z)$ and $\mathfrak{Q}_\nu^\mu(z)$, are equal to those conventionally used for $z \notin (-1, 1)$. The practice of using the *Fraktur* characters $\mathfrak{P}$ and $\mathfrak{Q}$ to denote the functions of type II follows Meixner and Schäfke [1].

Appendix A.1. Legendre functions

The Legendre functions of the first kind (of types I and II) are defined by

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \qquad \text{(type I)} \tag{A.1}$$

$$\mathfrak{P}_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \frac{(z+1)^{\mu/2}}{(z-1)^{\mu/2}} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \qquad \text{(type II)} \tag{A.2}$$

where $\Gamma(z)$ is the gamma function ([12], ch 6) and ${}_2F_1(a, b; c; z)$ is the Gaussian hypergeometric function ([12], ch 15). Note that these definitions differ only in their phase, and are trivially related:

$$\mathfrak{P}_\nu^\mu(z) = \frac{(1-z)^{\mu/2}}{(z-1)^{\mu/2}} P_\nu^\mu(z). \tag{A.3}$$

For noninteger μ the functions of the second kind are defined by

$$Q_\nu^\mu(z) = \frac{\pi \csc(\mu\pi)}{2} \left(\cos(\mu\pi) P_\nu^\mu(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(z)\right) \qquad \text{(type I)} \tag{A.4}$$

$$\mathfrak{Q}_\nu^\mu(z) = \frac{\pi\,\csc(\mu\pi)}{2}\,\mathrm{e}^{\mathrm{i}\mu\pi}\left(\mathfrak{P}_\nu^\mu(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\mathfrak{P}_\nu^{-\mu}(z)\right) \qquad \text{(type II)}. \tag{A.5}$$

The types I and II functions of the second kind are related by

$$\mathfrak{Q}_\nu^\mu(z) = \mathrm{e}^{\mathrm{i}\mu\pi}\frac{(z-1)^{\mu/2}}{(1-z)^{\mu/2}}\left(Q_\nu^\mu(z) + \frac{\pi\,\csc(\mu\pi)}{2}\left(\frac{(1-z)^\mu}{(z-1)^\mu} - \cos(\mu\pi)\right)P_\nu^\mu(z)\right) \tag{A.6}$$

$$\mathfrak{Q}_\nu^m(z) = (-1)^m\frac{(z-1)^{m/2}}{(1-z)^{m/2}}\left(Q_\nu^m(z) + \frac{\pi}{2}\frac{\sqrt{1-z}}{\sqrt{z-1}}P_\nu^m(z)\right) \tag{A.7}$$

where in the first equation μ is noninteger while in the second m is an integer.

The function $\mathfrak{Q}_\nu^\mu(z)$ has the following important hypergeometric representation:

$$\mathfrak{Q}_\nu^\mu(z) = \mathrm{e}^{\mathrm{i}\mu\pi}\frac{\sqrt{\pi}\,\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}z^{-\mu-\nu-1}(z+1)^{\mu/2}(z-1)^{\mu/2}$$
$$\times\,{}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \nu+\frac{3}{2}; \frac{1}{z^2}\right). \tag{A.8}$$

Because of the factor $\Gamma(\mu+\nu+1)$, this expansion diverges when $\mu+\nu = -1, -2, \ldots$ and hence

$$\left|Q_\nu^\mu(z)\right| \qquad \left|\mathfrak{Q}_\nu^\mu(z)\right| \to \infty \qquad \text{for} \quad \mu+\nu = -1, -2, \ldots \tag{A.9}$$

The following relations for $\nu \to -\nu-1$ are used in section 2.2:

$$Q_{-\nu-1}^\mu(z) = \csc((\mu-\nu)\pi)\left(\pi\cos(\mu\pi)\cos(\nu\pi)P_\nu^\mu(z) - \sin((\mu+\nu)\pi)Q_\nu^\mu(z)\right) \tag{A.10}$$

$$\mathfrak{Q}_{-\nu-1}^\mu(z) = \csc((\mu-\nu)\pi)\left(\pi\,\mathrm{e}^{\mathrm{i}\mu\pi}\cos(\nu\pi)\mathfrak{P}_\nu^\mu(z) - \sin((\mu+\nu)\pi)\mathfrak{Q}_\nu^\mu(z)\right). \tag{A.11}$$

In the *Mathematica* system the Legendre functions of types I and II are called 'type 2' and 'type 3' functions, respectively ('type 1' is a redundant variant of 'type 2' which is only defined for $|z| \leqslant 1$).

Appendix A.2. Spherical Bessel functions

The spherical Bessel function of the first kind is defined by

$$j_\nu(z) = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma\left(k+\nu+\frac{3}{2}\right)k!}\left(\frac{z}{2}\right)^{2k+\nu}. \tag{A.12}$$

The function of the second kind is defined in terms of the function of the first kind by

$$y_\nu(z) = -\sec(\nu\pi)(\sin(\nu\pi)j_\nu(z) + j_{-\nu-1}(z)). \tag{A.13}$$

The spherical Bessel functions are not implemented directly in the *Mathematica* system, but they can be computed by using their relation to the regular Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ ([12], ch 9), which are built-in:

$$j_\nu(z) = \sqrt{\frac{\pi}{2}}\frac{J_{\nu+\frac{1}{2}}(z)}{\sqrt{z}} \qquad y_\nu(z) = \sqrt{\frac{\pi}{2}}\frac{Y_{\nu+\frac{1}{2}}(z)}{\sqrt{z}}. \tag{A.14}$$

Appendix B. Flammer's spheroidal functions

In this appendix we give the essential relations between the spheroidal functions of Flammer [2] and those of Meixner [1]. Note that Flammer's functions are only defined for integer parameters m, n with $n \geqslant m \geqslant 0$.

Eigenvalue:

$$\lambda_{mn}(c) = \lambda_n^m(c) + c^2. \tag{B.1}$$

Angular functions:

$$S_{mn}(c,\eta) = \omega_{mn}(c)\mathrm{ps}_n^m(\eta;c) \tag{B.2}$$

$$S_{mn}^{(2)}(c,\eta) = \omega_{mn}(c)\mathrm{qs}_n^m(\eta;c). \tag{B.3}$$

where

$$\omega_{mn}(c) = \begin{cases} \dfrac{(-1)^{\frac{n-m}{2}}(m+n)!}{2^n\left(\frac{n-m}{2}\right)!\left(\frac{m+n}{2}\right)!}\dfrac{1}{\mathrm{ps}_n^m(0;c)} & n-m \text{ even} \\ \dfrac{(-1)^{\frac{n-m-1}{2}}(m+n+1)!}{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{m+n+1}{2}\right)!}\dfrac{1}{\mathrm{ps}_n^{m\prime}(0;c)} & n-m \text{ odd.} \end{cases} \tag{B.4}$$

Radial functions:

$$R_{mn}^{(1,2)}(c;\xi) = S_n^{m(1,2)}(\xi;c). \tag{B.5}$$

Appendix C. Symmetry relations for the spheroidal functions

The spheroidal wavefunctions satisfy a number of useful mathematical identities involving transformations of μ, ν and z, which they 'inherit' from properties of the Legendre and spherical Bessel functions. These were originally derived in a number of papers by Meixner (almost all of whose work was published in German—see [1] and references therein), but do not appear in the most popular references on spheroidal wavefunctions (e.g. [2, 12]), and may therefore be difficult to obtain for many contemporary readers. For the sake of completeness, we present in this appendix a concise summary of these identities, including certain special cases for integer parameters which are not trivially obtained from the general ones. We also present identities relating the angular and radial spheroidal functions, which are important for their computation. A more detailed discussion of the derivation of these identities can be found in [14].

It is important to note that all identities presented in this appendix are carefully derived so as to be valid throughout the complex z-plane, and in particular along branch cuts. These branch cuts are not arbitrary, but are inherent in the algebraic prefactors of the spheroidal wavefunctions, i.e. $(1+z)^{\mu/2}/(1-z)^{\mu/2}$ and $(1-1/z^2)^{\mu/2}$. This results in some unconventional expressions, such as $\sqrt{-z^2}/z$ or $(-\gamma z)^{\nu}(\gamma z)^{-\nu}$, which cannot be further simplified without changing the branch cut structure.

Appendix C.1. The transformation $\nu \to -\nu - 1$

Eigenvalue and radial normalization factor:

$$\lambda_{-\nu-1}^{\mu}(\gamma) = \lambda_{\nu}^{\mu}(\gamma) \qquad A_{-\nu-1}^{\mu}(\gamma) = A_{\nu}^{\mu}(\gamma). \tag{C.1}$$

Angular functions, general μ:

$$f_{-\nu-1}^{\mu}(z;\gamma) = f_{\nu}^{\mu}(z;\gamma) \qquad \text{for} \quad f = \mathrm{Ps}, \mathrm{ps} \tag{C.2}$$

$$\mathrm{qs}_{-\nu-1}^{\mu}(z;\gamma) = \csc((\mu-\nu)\pi)\left(\pi\cos(\mu\pi)\cos(\nu\pi)\mathrm{ps}_{\nu}^{\mu}(z;\gamma) - \sin((\mu+\nu)\pi)\mathrm{qs}_{\nu}^{\mu}(z;\gamma)\right) \tag{C.3}$$

$$\mathrm{Qs}_{-\nu-1}^{\mu}(z;\gamma) = \csc((\mu-\nu)\pi)\left(\pi\mathrm{e}^{\mathrm{i}\mu\pi}\cos(\nu\pi)\mathrm{Ps}_{\nu}^{\mu}(z;\gamma) - \sin((\mu+\nu)\pi)\mathrm{Qs}_{\nu}^{\mu}(z;\gamma)\right). \tag{C.4}$$

Angular functions of the second kind, integer m:

$$\mathrm{qs}_{-\nu-1}^{m}(z;\gamma) = \mathrm{qs}_{\nu}^{m}(z;\gamma) - \pi\cot(\nu\pi)\mathrm{ps}_{\nu}^{m}(z;\gamma) \tag{C.5}$$

$$\mathrm{Qs}_{-\nu-1}^{m}(z;\gamma) = \mathrm{Qs}_{\nu}^{m}(z;\gamma) - \pi\cot(\nu\pi)\mathrm{Ps}_{\nu}^{m}(z;\gamma). \tag{C.6}$$

Radial functions, general ν:

$$S_{-\nu-1}^{\mu\,(1)}(z;\gamma) = -\sin(\nu\pi)S_{\nu}^{\mu\,(1)}(z;\gamma) - \cos(\nu\pi)S_{\nu}^{\mu\,(2)}(z;\gamma) \tag{C.7}$$

$$S_{-\nu-1}^{\mu\,(2)}(z;\gamma) = \cos(\nu\pi)S_{\nu}^{\mu\,(1)}(z;\gamma) - \sin(\nu\pi)S_{\nu}^{\mu\,(2)}(z;\gamma). \tag{C.8}$$

Radial functions, integer n:

$$S_{-n-1}^{\mu\,(1)}(z;\gamma) = (-1)^{n+1}S_{n}^{\mu\,(2)}(z;\gamma) \tag{C.9}$$

$$S_{-n-1}^{\mu\,(2)}(z;\gamma) = (-1)^{n}S_{n}^{\mu\,(1)}(z;\gamma). \tag{C.10}$$

Appendix C.2. The transformation $\mu \to -\mu$

Eigenvalue:

$$\lambda_{\nu}^{-\mu}(\gamma) = \lambda_{\nu}^{\mu}(\gamma). \tag{C.11}$$

Angular functions, general μ:

$$\mathrm{ps}_{\nu}^{-\mu}(z;\gamma) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left(\cos(\mu\pi)\mathrm{ps}_{\nu}^{\mu}(z;\gamma) - \frac{2}{\pi}\sin(\mu\pi)\mathrm{qs}_{\nu}^{\mu}(z;\gamma)\right) \tag{C.12}$$

$$\mathrm{qs}_{\nu}^{-\mu}(z;\gamma) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left(\cos(\mu\pi)\mathrm{qs}_{\nu}^{\mu}(z;\gamma) + \frac{\pi}{2}\sin(\mu\pi)\mathrm{ps}_{\nu}^{\mu}(z;\gamma)\right) \tag{C.13}$$

$$\mathrm{Ps}_{\nu}^{-\mu}(z;\gamma) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left(\mathrm{Ps}_{\nu}^{\mu}(z;\gamma) - \frac{2}{\pi}\mathrm{e}^{-\mathrm{i}\mu\pi}\sin(\mu\pi)\mathrm{Qs}_{\nu}^{\mu}(z;\gamma)\right) \tag{C.14}$$

$$\mathrm{Qs}_{\nu}^{-\mu}(z;\gamma) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\mathrm{e}^{-2\mathrm{i}\mu\pi}\mathrm{Qs}_{\nu}^{\mu}(z;\gamma). \tag{C.15}$$

Angular functions, integer m:

$$f_{\nu}^{-m}(z;\gamma) = (-1)^{m}\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}f_{\nu}^{m}(z;\gamma) \qquad f = \mathrm{ps},\mathrm{qs} \tag{C.16}$$

$$f_{\nu}^{-m}(z;\gamma) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}f_{\nu}^{m}(z;\gamma) \qquad f = \mathrm{Ps},\mathrm{Qs}. \tag{C.17}$$

Radial functions:

$$S_{\nu}^{-\mu\,(1,2)}(z;\gamma) = S_{\nu}^{\mu\,(1,2)}(z;\gamma). \tag{C.18}$$

Appendix C.3. The transformation $z \to -z$

Angular functions of type I, general μ, ν:

$$\mathrm{ps}_\nu^\mu(-z;\gamma) = \cos((\mu+\nu)\pi)\mathrm{ps}_\nu^\mu(z;\gamma) - \frac{2}{\pi}\sin((\mu+\nu)\pi)\mathrm{qs}_\nu^\mu(z;\gamma) \tag{C.19}$$

$$\mathrm{qs}_\nu^\mu(-z;\gamma) = -\cos((\mu+\nu)\pi)\mathrm{qs}_\nu^\mu(z;\gamma) - \frac{\pi}{2}\sin((\mu+\nu)\pi)\mathrm{ps}_\nu^\mu(z;\gamma). \tag{C.20}$$

Angular functions of type II, general μ, ν and $z \notin (-1, 1)$:

$$\mathrm{Ps}_\nu^\mu(-z;\gamma) = \mathrm{e}^{\pi\nu\sqrt{-z^2}/z}\mathrm{Ps}_\nu^\mu(z;\gamma) - \frac{2}{\pi}\,\mathrm{e}^{-\mathrm{i}\mu\pi}\sin((\mu+\nu)\pi)\mathrm{Qs}_\nu^\mu(z;\gamma) \tag{C.21}$$

$$\mathrm{Qs}_\nu^\mu(z;\gamma) = -\mathrm{e}^{-\pi\nu\sqrt{-z^2}/z}\mathrm{Qs}_\nu^\mu(z;\gamma). \tag{C.22}$$

Angular functions, integer m, n:

$$\mathrm{ps}_n^m(-z;\gamma) = (-1)^{m+n}\mathrm{ps}_n^m(z;\gamma) \tag{C.23}$$

$$\mathrm{qs}_n^m(-z;\gamma) = (-1)^{m+n+1}\mathrm{qs}_n^m(z;\gamma) \tag{C.24}$$

$$\mathrm{Ps}_n^m(-z;\gamma) = (-1)^{n}\mathrm{Ps}_n^m(z;\gamma) \qquad z \notin (0,1) \tag{C.25}$$

$$\mathrm{Qs}_n^m(-z;\gamma) = (-1)^{n+1}\mathrm{Qs}_n^m(z;\gamma) \qquad z \notin (0,1). \tag{C.26}$$

Radial functions, general ν:

$$S_\nu^{\mu\,(1)}(-z;\gamma) = (-\gamma z)^{\nu}(\gamma z)^{-\nu}S_\nu^{\mu\,(1)}(z;\gamma) \tag{C.27}$$

$$S_\nu^{\mu\,(2)}(-z;\gamma) = -(-\gamma z)^{-\nu}(\gamma z)^{\nu}\big(S_\nu^{\mu\,(2)}(z;\gamma) + (1 + (-\gamma z)^{2\nu}(\gamma z)^{-2\nu})\tan(\nu\pi)S_\nu^{\mu\,(1)}(z;\gamma)\big). \tag{C.28}$$

Radial functions, integer n:

$$S_n^{\mu\,(1)}(-z;\gamma) = (-1)^{n}S_n^{\mu\,(1)}(z;\gamma) \tag{C.29}$$

$$S_n^{\mu\,(2)}(-z;\gamma) = (-1)^{n+1}S_n^{\mu\,(2)}(z;\gamma). \tag{C.30}$$

Appendix C.4. Relations between angular functions of types I and II

$$\mathrm{Ps}_\nu^\mu(z;\gamma) = \frac{(1-z)^{\mu/2}}{(z-1)^{\mu/2}}\mathrm{ps}_\nu^\mu(z;\gamma) \tag{C.31}$$

$$\mathrm{Qs}_\nu^\mu(z;\gamma) = \mathrm{e}^{\mathrm{i}\mu\pi}\frac{(z-1)^{\mu/2}}{(1-z)^{\mu/2}}\left(\mathrm{qs}_\nu^\mu(z;\gamma) + \frac{\pi\csc(\mu\pi)}{2}\left(\frac{(1-z)^{\mu}}{(z-1)^{\mu}} - \cos(\mu\pi)\right)\mathrm{ps}_\nu^\mu(z;\gamma)\right). \tag{C.32}$$

The second of these must be treated specially for integer m:

$$\mathrm{Qs}_\nu^m(z;\gamma) = (-1)^m\frac{(z-1)^{m/2}}{(1-z)^{m/2}}\left(\mathrm{qs}_\nu^m(z;\gamma) + \frac{\pi}{2}\frac{\sqrt{1-z}}{\sqrt{z-1}}\mathrm{ps}_\nu^m(z;\gamma)\right). \tag{C.33}$$

Table 1. Eigenvalues $\lambda_n^m(\gamma)$ for integer m, n and real γ^2.

m	n	γ	$-\lambda_n^m(\gamma)$	$\lambda_n^m(\mathrm{i}\gamma)$
0	0	10	90.77169 57027 50054 84898 77312	18.97205 60550 42243 81391 09191
0	0	100	9900.75189 88910 16747 44954 21523	198.99747 46340 82548 13572 48103
0	1	10	71.86653 62671 73272 18538 10249	18.97206 19762 54415 92684 71574
0	1	100	9701.75954 33440 82366 62256 40610	198.99747 46340 82548 13572 48103
1	1	10	89.71223 12326 08531 82924 20083	37.88064 98956 19453 22628 71049
1	1	100	9899.74682 23865 85061 62347 24354	397.98984 67939 13121 45974 40124
1	2	10	70.66108 19583 85518 52994 19784	37.88084 87977 73011 20481 64243
1	2	100	9700.74415 65958 58817 37915 37425	397.98984 67939 13121 45974 40124

Table 2. Eigenvalues $\lambda_\nu^\mu(\gamma)$ for complex μ, ν and γ. Here we use the abbreviation $\alpha = 1 + \mathrm{i}$.

μ	ν	γ	$\mathrm{Re}(\lambda_\nu^\mu(\gamma))$	$\mathrm{Im}(\lambda_\nu^\mu(\gamma))$
0	0	α	0.05947 27697 35031 26247 06156	−1.33717 48778 05399 97103 72378
0	0	10α	9.24076 62146 34603 35159 57442	−189.98934 85956 57553 67515 08696
0	α	1	0.50186 77624 67004 53075 16266	2.95073 69925 18211 20706 17897
0	10α	1	9.50003 16512 34204 66677 88169	209.99925 73181 59354 50064 18858
α	0	1	−0.90781 92346 93494 49431 33571	0.93747 61281 94795 84236 49580
10α	0	1	−13.78249 20414 53639 96320 69792	17.03738 91416 68651 13441 81797
α	α	α	1.14617 35587 36254 25050 29932	1.33182 58434 94567 67063 46083
10α	10α	10α	13.77544 66537 42879 55398 69299	14.13344 43105 19156 64488 99153

Table 3. Spheroidal joining factors, $K_n^m(\gamma)$, and radial normalization factors, $A_n^m(\gamma)$.

μ	ν	γ	$K_n^m(\gamma)$	$A_n^m(\gamma)$
0	0	10	428.00699 32832 74587 46082 31771	0.00092 59959 00168 65734 97377
0	0	10i	129.95559 15348 21883 48851 72999	4.35228 56879 68459 42426 84085
0	1	10	89.19337 15984 63365 35496 77125	0.00444 35150 58595 83160 08488
0	1	10i	225.08939 49188 27718 78315 26216i	2.51279 49340 42137 95801 16552
1	1	10	494.80660 08906 73485 53502 72883	−0.00008 45665 54309 12744 77141
1	1	10i	27.46214 85726 35368 05266 46285i	0.55989 62979 48596 23342 04583
1	2	10	43.80698 90817 46872 38955 24520	−0.00107 62847 63641 61970 54025
1	2	10i	−20.46826 39377 38829 05641 78445	0.75119 25147 80086 50861 25804

Appendix C.5. Relations between angular and radial functions

In this section we present the set of relations between the angular and radial functions which we use to compute the functions throughout the complex z-plane. They can be derived using (23) and the identities given in this appendix.

Radial functions, general μ, ν:

$$S_\nu^{\mu\,(1)}(z;\gamma) = K_\nu^\mu(\gamma)\frac{\sin((\mu-\nu)\pi)}{\pi}\frac{\mathrm{e}^{-\mathrm{i}(\mu+\nu)\pi}(1-1/z^2)^{\mu/2}(\gamma z)^\nu}{\gamma^\nu z^{\nu-\mu}(z-1)^{\mu/2}(z+1)^{\mu/2}}\mathrm{Qs}_{-\nu-1}^\mu(z;\gamma) \tag{C.34}$$

$$S_\nu^{\mu\,(2)}(z;\gamma) = \sec(\nu\pi)\big(S_{-\nu-1}^{\mu\,(1)}(z;\gamma) - \sin(\nu\pi)S_\nu^{\mu\,(1)}(z;\gamma)\big). \tag{C.35}$$

Table 4. Type I angular functions and their derivatives at $z = 0$.

m	n	γ	$\mathrm{ps}_n^m(0;\gamma)$	$\mathrm{ps}_{n+1}^{m}{}'(0;\gamma)$
0	0	10	$1.86950\,13198\,83220\,32378\,66070 \times 10^{0}$	$4.62218\,68979\,44534\,31859\,57783 \times 10^{0}$
0	0	10i	$8.13921\,06153\,91477\,31355\,92685 \times 10^{-4}$	$4.20017\,80506\,23196\,12220\,71385 \times 10^{-3}$
1	1	10	$1.52903\,37582\,54318\,09757\,33869 \times 10^{0}$	$-8.82749\,07181\,87103\,21096\,49776 \times 10^{0}$
1	1	10i	$-4.10717\,23604\,57252\,74666\,32257 \times 10^{-3}$	$-4.33152\,86911\,29750\,60250\,68055 \times 10^{-2}$
m	n	γ	$\mathrm{qs}_{n+1}^m(0;\gamma)$	$\mathrm{qs}_n^{m}{}'(0;\gamma)$
0	0	10	$-4.27174\,98257\,69345\,65574\,94192 \times 10^{-6}$	$4.58661\,56819\,97616\,23157\,52064 \times 10^{-7}$
0	0	10i	$-1.50330\,25515\,69445\,99770\,03079 \times 10^{3}$	$2.32730\,07180\,66502\,65903\,60560 \times 10^{4}$
1	1	10	$-9.27251\,18702\,47298\,25165\,14180 \times 10^{-6}$	$4.19596\,49830\,13982\,19782\,26061 \times 10^{-7}$
1	1	10i	$4.93493\,01484\,86571\,35347\,97135 \times 10^{2}$	$-2.89127\,68871\,67718\,83087\,43378 \times 10^{3}$

Table 5. Radial functions and their derivatives at $z = 1.005$.

m	n	γ	$S_n^{m(1)}(z;\gamma)$	$S_n^{m(1)\prime}(z;\gamma)$
2	2	1	$6.61191\,32248\,51537\,44227\,25009 \times 10^{-4}$	$1.32472\,88100\,07683\,20705\,27851 \times 10^{-1}$
2	2	2	$2.56592\,96586\,98996\,40081\,40566 \times 10^{-3}$	$5.12978\,72006\,11894\,29814\,83008 \times 10^{-1}$
3	2	3	$2.20653\,45978\,82418\,05038\,85691 \times 10^{-3}$	$4.42319\,54640\,28593\,94205\,30600 \times 10^{-1}$
3	2	4	$4.68276\,42681\,95501\,75619\,52436 \times 10^{-3}$	$9.34757\,21512\,11403\,78681\,71462 \times 10^{-1}$
m	n	γ	$S_n^{m(2)}(z;\gamma)$	$S_n^{m(2)\prime}(z;\gamma)$
2	2	1	$-3.74977\,22396\,54243\,54812\,78539 \times 10^{2}$	$7.57364\,90437\,91073\,13553\,02702 \times 10^{4}$
2	2	2	$-4.85222\,67972\,28220\,36109\,36955 \times 10^{1}$	$9.73698\,58589\,49359\,43573\,03506 \times 10^{3}$
3	2	3	$-3.74287\,18891\,97107\,67822\,75646 \times 10^{1}$	$7.56605\,12493\,58967\,24757\,30118 \times 10^{3}$
3	2	4	$-1.33399\,79013\,10628\,13090\,07387 \times 10^{1}$	$2.66253\,29643\,35609\,64101\,07459 \times 10^{3}$

Radial functions, integer m, n:

$$S_n^{m(1)}(z;\gamma) = K_n^m(\gamma)\frac{(1-1/z^2)^{m/2}z^m}{(z-1)^{m/2}(z+1)^{m/2}}\mathrm{Ps}_n^m(z;\gamma) \tag{C.36}$$

$$S_n^{m(2)}(z;\gamma) = \frac{(-1)^{m+1}}{\gamma K_n^{-m}(\gamma)A_n^m(\gamma)A_n^{-m}(\gamma)}\frac{(1-1/z^2)^{m/2}z^m}{(z-1)^{m/2}(z+1)^{m/2}}\mathrm{Qs}_n^m(z;\gamma). \tag{C.37}$$

Type II angular functions, general μ, ν:

$$\mathrm{Qs}_\nu^\mu(z;\gamma) = \frac{\pi\,\mathrm{e}^{\mathrm{i}(\mu+\nu)\pi}\csc((\mu+\nu)\pi)(z-1)^{\mu/2}(z+1)^{\mu/2}(\gamma z)^{\nu+1}}{K_{-\nu-1}^{\mu}(\gamma)(1-1/z^2)^{\mu/2}\gamma^{\nu+1}z^{\mu+\nu+1}}S_{-\nu-1}^{\mu(1)}(z;\gamma) \tag{C.38}$$

$$\mathrm{Ps}_\nu^\mu(z;\gamma) = \frac{\sec(\nu\pi)}{\pi}\,\mathrm{e}^{-\mathrm{i}\mu\pi}\left(\sin((\mu+\nu)\pi)\mathrm{Qs}_\nu^\mu(z;\gamma) - \sin((\nu-\mu)\pi)\mathrm{Qs}_{-\nu-1}^\mu(z;\gamma)\right). \tag{C.39}$$

Type II angular functions, integer m, n:

$$\mathrm{Ps}_n^m(z;\gamma) = \frac{1}{K_n^m(\gamma)}\frac{(1-1/z^2)^{m/2}z^m}{(z-1)^{m/2}(z+1)^{m/2}}S_n^{m(1)}(z;\gamma) \tag{C.40}$$

$$\mathrm{Qs}_n^m(z;\gamma) = (-1)^{m+1}\gamma K_n^{-m}(\gamma) A_n^m(\gamma) A_n^{-m}(\gamma) \frac{(z-1)^{m/2}(z+1)^{m/2}}{(1-1/z^2)^{m/2} z^m} S_n^{m\,(2)}(z;\gamma). \tag{C.41}$$

Relations involving the type I angular functions, $\mathrm{ps}_\nu^\mu(z;\gamma)$ and $\mathrm{qs}_\nu^\mu(z;\gamma)$, can be obtained from the last four equations by using (C.31)–(C.33).

Appendix D. Tables of numerical values

In this appendix we present a set of numerical values, to 25 digits of precision, for all of Meixner's spheroidal functions. The intention is to provide enough values and to sufficient precision to facilitate comparison with any future software implementation. Tables 1 and 2 contain the eigenvalues $\lambda_\nu^\mu(\gamma)$ for integer and complex parameters, respectively; table 3 contains the joining and normalization factors $K_\nu^\mu(\gamma)$ and $A_\nu^\mu(\gamma)$; finally, tables 4 and 5 contain values of the angular and radial functions and their derivatives. Values of the type II functions, as well as all of Flammer's functions, are not given, since they can be easily found from those presented here.

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